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A method to generate algebraically special pure radiation field solutions from the vacuum

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Abstract. A method is proposed to generate solutions of Einstein's field equations for pure radiation fields from the vacuum. The method applies to all algebraically special diverging vacuum metrics, reducing the field equations to a single partial differential equation for one real function. This equation becomes linear if additional assumptions concerning the vacuum metric are made. Several classes of explicit solutions have been found.

1. The problem

In this paper we consider algebraically special pure radiation fields. More precisely, we are interested in exact solutions of Einstein's equations

$$R_{ab} = \kappa\phi^2 k_a k_b, \qquad k_a k^a = 0, \qquad \phi^2 > 0, \tag{1.1}$$

(R_{ab} Ricci tensor, κ gravitational constant) which have the following properties: The null vector k^a is (i) a multiple eigenvector of the Weyl tensor and is (ii) geodetic and shear free, but diverging:

$$\begin{aligned} &k_{[n}k_{a];b}k^b = 0, \qquad k_{(a;b)}k^{a;b} - (k^a{}_{;a})^2/2 = 0, \\ &k^a{}_{;a} = 2\theta \neq 0. \end{aligned} \tag{1.2}$$

The method we shall develop applies also to all solutions with a non-twisting null congruence, but as in that case it is more or less trivial to generate radiation field solutions we shall concentrate on the twisting case:

$$2\omega^2 = k_{[a;b]}k^{a;b} \neq 0. \tag{1.3}$$

It has been shown (Robinson and Robinson 1969; Robinson, Schild and Strauss 1969) that with the assumptions stated above the metric can be written in the form

$$\begin{aligned} &ds^2 = 2\omega^1\omega^2 - 2\omega^3\omega^4 \\ &\omega^1 = \bar{\omega}^2 = -d\zeta/\bar{\rho}P(\zeta, \bar{\zeta}, u) \\ &\omega^3 = -k_n\, dx^n = du + L(\zeta, \bar{\zeta}, u)\, d + \bar{L}\, d\bar{\zeta} \\ &\omega^4 = dr + W(\zeta, \bar{\zeta}, r, u)\, d + \bar{W}\, d\bar{\zeta} + H(\zeta, \bar{\zeta}, r, u)\omega^3 \end{aligned} \tag{1.4}$$

with

$$\begin{aligned}
&\rho^{-1} = -(\theta + \mathrm{i}\omega)^{-1} = -(r + \mathrm{i}\Sigma), \qquad 2\mathrm{i}\Sigma = P^2(\bar{\partial} L - \partial \bar{L}) \\
&H = -r(\ln P)_{,u} - [m(\zeta, \bar{\zeta}, u)r + M\Sigma]/(r^2 + \Sigma^2) + K/2 \\
&K = 2P^2 \operatorname{Re}[\partial(\bar{\partial} \ln P - \bar{L}_{,u})], \qquad W = \rho^{-1} L_{,u} + \mathrm{i}\partial\Sigma \\
&\partial \equiv \partial_\zeta - L\partial_u, \qquad \bar{\partial} \equiv \partial_{\bar{\zeta}} - \bar{L}\partial_u
\end{aligned} \tag{1.5}$$

and

$$M(\zeta, \bar{\zeta}, u) = \Sigma K + P^2 \operatorname{Re}[\partial\bar{\partial}\Sigma - 2\bar{L}_{,u}\partial\Sigma - \Sigma\partial_u\partial L]. \tag{1.6}$$

The remaining field equations then read

$$(3L_{,u} - \partial)(m + \mathrm{i}M) = 0 \tag{1.7}$$

$$\begin{aligned}
&P^4(\partial - 2L_{,u} + 2\partial \ln P)\partial I - P^3[P^{-3}(m + \mathrm{i}M)]_{,u} = n^2(\zeta, \bar{\zeta}, u)/2 \\
&I \equiv \bar{\partial}(\bar{\partial} \ln P - \bar{L}_{,u}) + (\bar{\partial} \ln P - \bar{L}_{,u})^2,
\end{aligned} \tag{1.8}$$

where n^2 is related to the radiation field (represented by ϕ^2) by

$$\phi^2 = n^2(\zeta, \bar{\zeta}, u)\rho\bar{\rho}. \tag{1.9}$$

The main content of the equations (1.4)–(1.9) can be summarised as follows: If one takes the affine parameter r of the null congruence as the coordinate x^3, $k^a = (0, 0, 1, 0)$, then two complex coordinates $x^1 = \zeta$, $x^2 = \bar{\zeta}$ and the real (retarded time) coordinate $x^4 = u$ can be introduced, as in (1.4). Equations (1.5), (1.6), (1.8) and (1.9) show that the radial (r) dependence of all metric functions and of the radiation field can be explicitly determined, and that all remaining functions of ζ, $\bar{\zeta}$ and u can be given in terms of the two real functions $P(\zeta, \bar{\zeta}, u)$ and $m(\zeta, \bar{\zeta}, u)$ and of the complex function $L(\zeta, \bar{\zeta}, u)$. The field equation (1.8) can be considered as giving the definition of n^2 in terms of P, m and L, the essential condition being that the left hand side must be positive. Thus the complex equation (1.7), together with the definition (1.6) of M in terms of P and L, is *the* field equation for the case considered here. To find an exact solution means to find functions P, m and L which satisfy the system (1.6)–(1.7) and give a non-negative n^2 from (1.8).

The coordinates $(\zeta, \bar{\zeta}, r, u)$ are not uniquely defined by (1.4). Besides the trivial change $r' = r + r^0(\zeta, \bar{\zeta}, u)$ of the origin of the affine parameter, we are free to make the transformations

$$\zeta' = f(\zeta), \qquad u' = F(u, \zeta, \bar{\zeta}), \qquad r' = F_{,u}^{-1} r, \tag{1.10}$$

which induce the transformations

$$\begin{aligned}
&\rho' = F_{,u}\rho, \qquad P' = |f'|F_{,u}^{-1}P, \qquad n^{2'} = F_{,u}^{-4} n^2, \\
&L' = f'^{-1}(F_{,u}L - F_{,\zeta}), \qquad (m + \mathrm{i}M)' = F_{,u}^{-3}(m + \mathrm{i}M).
\end{aligned} \tag{1.11}$$

These transformations can be used to simplify the field equations and/or the solutions They show that from the four real functions entering the field equations at least one can be prescribed.

For the vacuum case, we simply have to cancel n^2 in equations (1.8).

2. Some remarks concerning known classes of solutions

So far the problem to find solutions of the field equations (1.7)–(1.8) for pure radiation has been attacked in a systematic way along three lines. We will discuss them briefly.

If the null congruence is non-twisting ($\omega = 0$), then by a coordinate transformation (1.10) $L = \Sigma = M = 0$ can be achieved, and the field equation (1.7) gives $m = m(u)$. No further differential equation needs to be satisfied, we must only choose $P(\zeta, \bar{\zeta}, u)$ such that n^2 becomes positive. A well-known example of this class is Vaidya's radiating Schwarzschild-metric (Vaidya 1951).

Vaidya and co-workers (Vaidya 1974, Vaidya and Patel 1973) tried to find solutions of the Kerr–Schild class, i.e. solutions with a metric tensor of the form $g_{mn} = \eta_{mn} + V(x^i)k_m k_n$, η_{mn} representing a flat space–time. The emphasis here was on the construction of a radiating Kerr metric.

Hughston (1971) gave a method to generate radiation fields from vacuum by leaving P and L fixed but changing the function m. A coupled system of equations for the new m has to be solved. Unfortunately, this method has to assume $L_{,u} \neq 0$ for the vacuum solution to start with, and from all vacuum solutions known so far it can be applied only to flat space–time (yielding a non-trivial class of pure radiation fields).

3. The generation procedure

The main idea of the procedure proposed in this paper is as follows: As already stated, the essential field equation is (1.7), which is the same as in the vacuum case. Suppose now that we already have a vacuum (or radiation field) solution which satisfies (1.7). Then the easiest way to obtain a (new) radiation field solution is to leave unchanged all functions (L, m, M) which enter (1.7), and to change only P to obtain a non-zero or changed n^2 from (1.8). Equation (1.6) shows that M depends on P, so this trick will work only if P is chosen such that M does *not* change. This condition is in fact a differential equation for the function P, but in contrast to the original coupled system (1.7) of two differential equations for four real functions we now have to deal only with a single differential equation for one real function. This differential equation will become linear if certain assumptions for the initial vacuum metric are made.

Theorem: If (L^0, P^0, m^0, M^0) is an algebraically special diverging vacuum or pure radiation metric which satisfies (1.4)–(1.9), then

$$L = L^0, \qquad M = M^0, \qquad m = m^0, \qquad P = P^0\pi(\zeta, \bar{\zeta}, u), \tag{3.1}$$

is a pure radiation field exactly if the real function π obeys

$$\begin{aligned} &2(P^0)^2\pi^3[\partial(\Sigma^0\bar{\partial}\pi) + \bar{\partial}(\Sigma^0\partial\pi) - \Sigma^0(\bar{L}_{,u}\partial\pi + L_{,u}\bar{\partial}\pi) + M^0(\pi^4 - 1)] = 0, \\ &2\mathrm{i}\,\Sigma^0 = (P^0)^2(\bar{\partial}L - \partial\bar{L}), \qquad \partial = \partial_\zeta - L\partial_u. \end{aligned} \tag{3.2}$$

The new solution is twisting if the original one is, i.e. for $\Sigma^0 \neq 0$. It is non-vacuum if (1.8) gives a non-zero (positive) n^2.

Proof. The proof of the theorem is simple and achieved by inserting (3.1) into the definition (1.6) of M. The differential equation (3.2) for π becomes linear if M^0

vanishes. All examples given below belong to this class, i.e. they are solutions of

$$\partial(\Sigma^0\bar{\partial}\pi)+\bar{\partial}(\Sigma^0\partial\pi)-\Sigma^0(\bar{L}_{,u}\partial\pi+L_{,u}\bar{\partial}\pi)=0. \tag{3.3}$$

4. Examples

(i) Independent of the properties of the initial vacuum solution (with $M^0=0$)

$$\pi=\text{constant}$$

is always a solution of (3.3). Unfortunately there seems to be no vacuum solution for which $[P^{-3}(m+\mathrm{i}M)]_{,u}$ is non-zero if M vanishes, so that n^2 remains zero and no radiation field can be generated.

(ii) In principle the technique of generation can be applied to the class of vacuum solutions characterised by $L_{,u}=P_{,u}=(m+\mathrm{i}M)_{,u}=0$, $\partial I\neq 0$. This class has been found by Robinson and Robinson (1969) and contains some twisting type III solutions (Robinson 1975, Held 1974). A simple solution of this kind is

$$L=\mathrm{i}a(\zeta-\bar{\zeta})(\zeta+\bar{\zeta})^{(\sqrt{13}-5)/2},\qquad P=(\zeta+\bar{\zeta})^{3/2},\qquad a\text{ real}. \tag{4.1}$$

A solution $\pi=\pi(\zeta+\bar{\zeta})$ can be easily constructed, but the corresponding n^2 is negative,

$$\pi=b(\bar{\zeta}+\zeta)^{(1-\sqrt{13})/2},\qquad \kappa n^2=b^4(27\sqrt{13}-57)(\zeta+\bar{\zeta})^{4-2\sqrt{13}} \tag{4.2}$$

and also $\pi=\mathrm{i}b(\zeta-\bar{\zeta})$ leads to a negative n^2 (for $\zeta+\bar{\zeta}>0$).

(iii) The Kerr–Schild class of vacuum solutions is given by

$$L^0=l(\bar{\zeta})(P^0)^{-2},\qquad P^0=\alpha\zeta\bar{\zeta}+\beta\zeta+\bar{\beta}\bar{\zeta}+\delta,\qquad m^0=\text{constant},\qquad M^0=0, \tag{4.3}$$

compare Trim and Wainwright (1974) and Debney *et al* (1969). Explicit radiation field solutions, i.e. solutions of the corresponding differential equation for π, have been found for several sub-cases. Note that in the case of (4.3) equation (1.8) simply reads

$$P^3\partial\partial\bar{\partial}\bar{\partial}P-P^2(\partial\partial P)(\bar{\partial}\bar{\partial}P)+3m^0\partial_u\ln P=\kappa n^2/2, \tag{4.4}$$

which shows that for $\pi_{,u}\neq 0$, n^2 can be made positive by choice of m^0 for at least some region of space–time.

If we assume π to be a function only of u we obtain

$$\pi(u)=bu,\qquad L^0=(c_1\bar{\zeta}^2+c_2\bar{\zeta}+c_3)^{1/2},\qquad P^0=1,\qquad m^0=\text{constant},\qquad M=0$$
$$b,\ c_i\text{ constants},\qquad b\text{ and }c_1\text{ real} \tag{4.5}$$

and

$$\begin{aligned}&\pi(u)=bu, &&L^0=-\mathrm{i}a^0\bar{\zeta}(P^0)^{-2}\\ &P^0=1+K\zeta\bar{\zeta}/2, &&K=0,\pm 1, \qquad a^0,\ b\text{ real constants}\end{aligned} \tag{4.6}$$

as radiating solutions. In both cases

$$\kappa n^2/2=3m^0/u \tag{4.7}$$

holds, i.e. n^2 can be made positive by choice of m^0.

The solution (4.6) with $K=1$ is a radiating Kerr metric (asymptotically flat and again of the Kerr–Schild class), first given by Kramer (1972) and in transformed coordinates by Goodinson (1972). To obtain a more familiar form of this solution we perform a coordinate transformation (1.10) with $f = \zeta$ and $F = bu^2/2$, which transforms $P = \pi P^0$ back into P^0. The result is

$$L = -ia\bar{\zeta}P^{-2}, \qquad P = 1 + \zeta\bar{\zeta}/2,$$
$$m = m^0(2bu)^{-3/2}, \qquad a = a^0(2bu)^{1/2}. \tag{4.8}$$

This differs from the Kerr metric exactly by a (special) time (t) dependence of the mass parameter m and the Kerr parameter a.

If we assume axial symmetry, then we have to solve

$$L^0 = -ia\bar{\zeta}(P^0)^{-2}, \qquad P^0 = 1 + K\zeta\bar{\zeta}/2, \qquad K = 0, \pm 1, \qquad a \text{ real}, \qquad m^0 = \text{constant}$$
$$(\Sigma^0 \zeta\bar{\zeta}\pi')' + \Sigma^0 a^2 \zeta\bar{\zeta}(P^0)^{-4}\ddot{\pi} = 0, \qquad \pi = \pi(\zeta\bar{\zeta}, u). \tag{4.9}$$

For $K = 0$ ($P = \pi$) the general solution of (4.9) is a superposition (with different α) of

$$\pi(\zeta\bar{\zeta}, u) = (b_1 u + b_2)(b_3 \ln \zeta\bar{\zeta} + b_4)$$
$$\pi(\zeta\bar{\zeta}, u) = [b_1 \exp(\alpha u) + b_2 \exp(-\alpha u)]J_0(a\alpha\zeta\bar{\zeta}), \qquad \alpha \neq 0, \tag{4.10}$$

J_0 being Bessel functions. For a fixed $\alpha \neq 0$, equation (4.4) shows that n^2 can be made positive by choice of m^0, b_1, b_2 and α.

For $K \neq 0$, besides (4.6) also

$$\pi(\zeta\bar{\zeta}) = a_1[\ln(K\zeta\bar{\zeta}/2) - 2\ln(1 - K\zeta\bar{\zeta}/2)] + a_2 \tag{4.11}$$

is a solution.

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